

QUANTUM NONABELIAN MONOPOLES

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We discuss quantum mechanical and topological aspects of nonabelian monopoles.
Related recent results on nonabelian vortices are also mentioned.

1. Prologue

There are several reasons to be interested in *quantum, nonabelian* monopoles. First, if confinement of QCD is a sort of dual superconductor, it is likely to be one of nonabelian variety. Then the effective degrees of freedom involve nonabelian, and not, better understood abelian monopoles. Second, the phenomenon of confinement has to do with fully quantum mechanical, and not semi-classical, behavior of the monopoles. Thirdly, the very concept of nonabelian monopoles is, as we shall see, intrinsically quantum mechanical, in contrast to that of the 't Hooft-Polyakov monopole carrying an abelian charge only. A semi-classical consideration only might easily lead us astray. Finally, some recent developments on nonabelian BPS *vortices* provide further hints on the subtle nature of nonabelian monopoles and related dual gauge transformations. These considerations are sufficient motivations to give a renewed look on the topological as well as dynamical aspects of these soliton states, in particular in relation to $N = 2$ gauge theories.

2. Confinement in $SU(N)$ YM Theory

The test charges in $SU(N)$ YM theory take values in $(Z_N^{(M)}, Z_N^{(E)})$ where Z_N is the center of $SU(N)$ and $Z_N^{(M)}, Z_N^{(E)}$ refer to the magnetic and electric center charges. $(Z_N^{(M)}, Z_N^{(E)})$ classification of phases follows^{1,2}. Namely,

- (1) If a field with $x = (a, b)$ condenses, particles $X = (A, B)$ with

$$\langle x, X \rangle \equiv a B - b A \neq 0 \pmod{N}$$

are confined. (e.g. $\langle \phi_{(0,1)} \rangle \neq 0 \rightarrow$ Higgs phase.)

- (2) Quarks are confined if some magnetically charged particle χ condenses, $\langle \chi_{(1,0)} \rangle \neq 0$.
- (3) In the softly broken $N = 4$ (to $N = 1$) theory (often referred to as $N = 1^*$) all different types of massive vacua, related by $SL(2, Z)$, appear; the chiral condensates in each vacua are known.
- (4) *Confinement index* ³ is equal to the smallest possible $r \in Z_N^{(E)}$ for which Wilson loop displays no area law. For instance, for $SU(N)$ YM, $r = N$ in the vacuum with complete confinement; $r = 1$ in the totally Higgs vacuum, etc.
- (5) In softly broken $N = 2$ gauge theories, dynamics turns out to be particularly transparent.

We are particularly interested in questions such as: What is χ in QCD? How do they interact? Is chiral symmetry breaking related to confinement?

A familiar idea is that the ground state of QCD is a dual superconductor ². Although there exist no elementary nor soliton monopoles in QCD, monopoles can be detected as topological singularities (lines in $4D$) of Abelian gauge fixing, $SU(3) \rightarrow U(1)^2$, as suggested by 't Hooft. Alternatively, one can assume that certain configurations close to the Wu-Yang monopole ($SU(2)$)

$$A_\mu^a \sim (\partial_\mu n \times n)^a + \dots, \quad n(r) = \frac{r}{r} \Rightarrow A_i^a = \epsilon_{aij} \frac{r^j}{r^3}$$

dominate ⁴.

Although there is some evidence in lattice QCD ⁵ for “Abelian dominance”, there remain several questions to be answered. Do abelian monopoles carry flavor? What is L_{eff} ? What about the gauge dependence of such abelian gauge-fixed action? Most significantly, does dynamical $SU(N) \rightarrow U(1)^{N-1}$ breaking occur? That would imply a richer spectrum of mesons ($T_1 \neq T_2$, etc) not seen in Nature and not expected in QCD. Both in Nature and presumably in QCD there is only one “meson” state, $\sum_{i=1}^N |q_i \bar{q}_i\rangle$, i.e., 1 state vs $[\frac{N}{2}]$ states. Note that it is not sufficient to assume the symmetry breaking $SU(N) \rightarrow U(1)^{N-1} \times Weyl$ symmetry, with an extra discrete symmetry, to solve the problem: the multiplicity would be wrong. If nonabelian degrees of freedom are important, after all, how do they manifest themselves?

3. “Semiclassical” Nonabelian Monopoles

Let us review briefly the standard results about nonabelian monopoles⁹⁻¹⁸. One is interested in a system with gauge symmetry breaking

$$G \xrightarrow{\langle \phi \rangle \neq 0} H$$

where H is non abelian. Asymptotic behavior of scalar and gauge fields (for a finite action) are:

$$\phi \sim U \cdot \langle \phi \rangle \cdot U^{-1} \sim \Pi_2(G/H) = \Pi_1(H);$$

$$A_i^a \sim U \cdot \partial_i U^\dagger \rightarrow F_{ij} \sim \epsilon_{ijk} \frac{r_k}{r^3} (\beta \cdot \mathbf{H}), \quad H_i \in \text{Cartan S.A. of } G.$$

Topological Quantization then leads to

$$2\alpha \cdot \beta \in \mathbb{Z}, \quad \text{cfr. } 2g_e g_m = n \quad (1)$$

$$\beta_i = \text{weight vectors of } \tilde{H} \quad (= \text{dual of } H),$$

namely, the nonabelian monopoles are characterized by the weight vectors of the dual group \tilde{H} . A general formula for the semiclassical monopole solutions (set $\langle \phi_0 \rangle = \mathbf{h} \cdot \mathbf{H}$) is given in terms of various broken $SU(2)$ subgroups,

$$S_1 = \frac{1}{\sqrt{2}\alpha^2} (E_\alpha + E_{-\alpha}); \quad S_2 = -\frac{i}{\sqrt{2}\alpha^2} (E_\alpha - E_{-\alpha}); \quad S_3 = \alpha^* \cdot \mathbf{H};$$

the nonabelian monopoles are basically an embedding of the 't Hooft-Polyakov monopoles⁸ in such $SU(2)$ subgroups:

$$A_i(\mathbf{r}) = A_i^a(\mathbf{r}, \mathbf{h} \cdot \alpha) S_a; \quad \phi(\mathbf{r}) = \chi^a(\mathbf{r}, \mathbf{h} \cdot \alpha) S_a + [\mathbf{h} - (\mathbf{h} \cdot \alpha) \alpha^*] \cdot \mathbf{H}, \quad (2)$$

where $(\alpha^* \equiv \alpha/(\alpha \cdot \alpha))$

$$A_i^a(\mathbf{r}) = \epsilon_{aij} \frac{r^j}{r^2} A(r); \quad \chi^a(\mathbf{r}) = \frac{r^a}{r} \chi(r), \quad \chi(\infty) = \mathbf{h} \cdot \alpha.$$

The mass and $U(1)$ flux can be easily calculated:

$$M = \int d\mathbf{S} \cdot \text{Tr } \phi \mathbf{B}, \quad \mathbf{B} = \frac{r_i(\mathbf{S} \cdot \mathbf{r})}{r^4} = \frac{\mathbf{r} S_3}{r^3} = \frac{\mathbf{r}}{r^3} \alpha^* \cdot \mathbf{H};$$

$U(1)$ flux (for instance, for $SU(N+1) \rightarrow SU(N) \times U(1)$) is

$$F_m = \int_{S^2} d\mathbf{S} \cdot \frac{\text{Tr } \phi \mathbf{B}}{\frac{1}{\sqrt{2}}(\text{Tr } \phi^2)^{1/2}} \equiv 4\pi g_m = 2\pi \cdot \sqrt{\frac{2(N+1)}{N}} \quad (3)$$

Example of dual groups (defined by $\alpha \Leftrightarrow \alpha^*$) are:

$$\begin{array}{c}
\hline
SU(N)/Z_N \quad \Leftrightarrow \quad SU(N) \\
SO(2N) \quad \Leftrightarrow \quad SO(2N) \\
SO(2N+1) \quad \Leftrightarrow \quad USp(2N) \\
\hline
\end{array}$$

4. Some Examples

The simplest system with nonabelian monopoles involves the gauge symmetry breaking,

$$SU(3) \xrightarrow{\langle \phi \rangle} \frac{SU(2) \times U(1)}{\mathbb{Z}_2}, \quad \langle \phi \rangle = \begin{pmatrix} v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -2v \end{pmatrix}$$

The monopole solutions are

$$\phi(\mathbf{r}) = \begin{pmatrix} -\frac{1}{2}v & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & -\frac{1}{2}v \end{pmatrix} + 3v \hat{S} \cdot \hat{r} \phi(r), \quad \vec{A}(\mathbf{r}) = \hat{S} \wedge \hat{r} A(r),$$

$\phi(r)$ and $A(r)$ are BPS 't Hooft's functions with $\phi(\infty) = 1$, $\phi(0) = 0$, $A(\infty) = -1/r$, where \hat{S} is an $SU(2)$ subgroup

$$S^1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad S^2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}; \quad S^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

or an analogous one in the $(2-3)$ rows and columns. So in this case there are two *degenerate* $SU(3)$ solutions.

The generalization to the case with symmetry breaking

$$SU(N+1) \rightarrow \frac{SU(N) \times U(1)}{\mathbb{Z}_N},$$

$$\langle \phi \rangle = \begin{pmatrix} v & 0 & \dots & 0 \\ 0 & v & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -Nv \end{pmatrix} = \begin{pmatrix} v \cdot \mathbf{1}_{N \times N} & \\ & -Nv \end{pmatrix},$$

is straightforward. Consider a broken $SU(2)$, S_i living in $(1, N+1)$ rows/columns: then

$$\phi = \begin{pmatrix} -\frac{N-1}{2}v & 0 & \dots & 0 & 0 \\ 0 & v & 0 & \dots & 0 \\ 0 & 0 & v & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\frac{N-1}{2}v \end{pmatrix} + (N+1)v(\vec{S} \cdot \hat{r})\phi(r),$$

$$\vec{A}(r) = \vec{S} \wedge \hat{r} \frac{A(r)}{g}$$

gives a monopole solution of $SU(N+1)$ equations of motion. By considering various $SU(2)$ subgroups living in $(i, N+1)$ rows/columns, $i = 1, 2, \dots, N$, one is led to N degenerate solutions.

5. Homotopy Groups in Systems $G \rightarrow H$

Let us consider now the relevant homotopy groups. The short exact sequence

$$0 \rightarrow \pi_2(G/H) \xrightarrow{f} \pi_1(H) \rightarrow \pi_1(G) \rightarrow 0.$$

tells us that regular (BPS) monopoles represent $\pi_2(G/H) \subset \pi_1(H) \subset \pi_1(G)$. Alternatively (Coleman) one can say that regular monopoles correspond to the kernel of mapping $\pi_1(H) \rightarrow \pi_1(G)$. In general, BPS monopoles belong to a k th tensor irrep of \tilde{H} , $k \in \pi_1(H)$. The relation between 't Hooft-Polyakov (regular) monopoles and Dirac (singular) monopoles is illustrated in Figure 1, which schematically represents the exact sequence above.

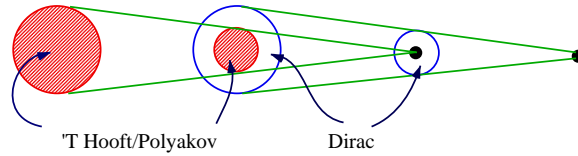


Figure 1.

6. Monopoles are multiplets of \tilde{H}

A crucial fact for us is that *monopoles are multiplet of \tilde{H}* and not of the original gauge group H . This is most clearly seen in the case of $USp(2N+2) \rightarrow USp(2N) \times U(1)$ where we find $2N+1$ degenerate monopoles (of $U\tilde{Sp}(2N) = SO(2N+1)$!), or in the system $SO(2n+3) \rightarrow SO(2n+1) \times U(1)$, where the multiplicity of degenerate monopoles is $2N$ (a right number for the fundamental representation of $\tilde{SO}(2N+1) = USp(2N)$.)

We have recently re-examined the possible irreducible representation (of the dual group \tilde{H}) to which monopoles belong, in various cases. The results are shown in Table 1 taken from ¹⁹.

G	H	\tilde{H}	Irrep	$U(1)$
$SU(N+1)$	$\frac{SU(N) \times U(1)}{\mathbb{Z}_N}$	$SU(N) \times U(1)$	N	$1/N$
$* SU(N)$	$\frac{SU(r) \times U(1)^{N-r+1}}{\mathbb{Z}_r}$	$SU(r) \times U(1)^{N-r+1}$	r	$1/r$
$USp(2N+2)$	$USp(2N) \times U(1)$	$SO(2N+1) \times U(1)$	$2N+1$	1
$* USp(2N+2)$	$\frac{SU(r) \times U(1)^{N-r}}{\mathbb{Z}_r}$	$SU(r) \times U(1)^{N-r+1}$	r	$1/r$
$SO(2N+3)$	$SO(2N+1) \times U(1)$	$USp(2N) \times U(1)$	$2N$	1
$SO(2N+2)$	$SO(2N) \times U(1)$	$SO(2N) \times U(1)$	$2N$	1
$USp(2N)$	$\frac{SU(N) \times U(1)}{\mathbb{Z}_N}$	$SU(N) \times U(1)$	N	$1/N$
$SO(2N)$	$\frac{SU(N) \times U(1)}{\mathbb{Z}_N}$	$SU(N) \times U(1)$	$\frac{N(N-1)}{2}$	$2/N$
$* SO(2N)$	$\frac{SU(r) \times U(1)^{N-r+1}}{\mathbb{Z}_r}$	$SU(r) \times U(1)^{N-r+1}$	r	$1/r$
$SO(2N+1)$	$\frac{SU(N) \times U(1)}{\mathbb{Z}_N}$	$SU(N) \times U(1)$	$\frac{N(N+1)}{2}$	$2/N$
$* SO(2N+1)$	$\frac{SU(r) \times U(1)^{N-r+1}}{\mathbb{Z}_r}$	$SU(r) \times U(1)^{N-r+1}$	r	$1/r$
$SU(N+M)$	$\frac{SU(N) \times SU(M) \times U(1)}{\mathbb{Z}_k}$	$SU(N) \times SU(M) \times U(1)$	(N, \bar{M})	$1/k$
$SO(2N+2M)$	$SO(2N) \times U(M)$	$SO(2N) \times U(M)$	$(2N, M)$	$1/M$
$SO(2N+2M+1)$	$SO(2N+1) \times U(M)$	$USp(2N) \times U(M)$	$(2N, M)$	$1/M$
$USp(2N+2M)$	$USp(2N) \times U(M)$	$SO(2N+1) \times U(M)$	$(2N+1, M)$	$1/M$

7. Why Nonabelian Monopoles are Intrinsically Quantum Mechanical

Nonabelian monopoles turn out to be essentially quantum mechanical. In fact, finding semiclassical degenerate monopoles, as reviewed above, is not sufficient for us to conclude that they form a multiplet of \tilde{H} , as H can break itself dynamically at lower energies and break the degeneracy among the monopoles. We must ensure that this does not take place. Nonabelian monopoles are in this sense never really semi-classical, even if $\langle \phi \rangle \gg \Lambda_H$: (e.g., Pure $N=2$, $SU(3)$).

In this connection, there is a famous “no go theorem” which states that there are no “colored dyons”¹⁶. For instance, in the background of the monopole arising from the breaking $\frac{SU(3)}{SU(2) \times U(1)}$, no global T^1, T^2, T^3 isomorphic to $SU(2)$ can be shown to exist. These results have somewhat obscured the whole issue of nonabelian monopoles for some time. Do they not exist? Are they actually inconsistent? The way out of this impasse is actually very simple: nonabelian monopoles are multiplets of the dual \tilde{H} group, and the results of¹⁶ does not exclude existence of sets of monopoles transforming as members of a dual multiplet (even if at present the explicit form of such nonlocal transformations are not known; see however below).

Nevertheless, the no-go theorem implies that the true gauge group of the system is not

$$G_{gauge} \neq H \otimes \tilde{H}$$

as sometimes suggested, but H or \tilde{H} or something else, according to which degrees of freedom are effectively present. (See also ¹²).

8. Phases of Softly Broken $N = 2$ Gauge Theories

Fully quantum mechanical results about the phases of $SU(n_c)$, $USp(2n_c)$ and $SO(n_c)$ theories with n_f hypermultiplets (quarks), perturbed by the superpotential

$$W(\phi, Q, \tilde{Q}) = \mu \text{Tr} \Phi^2 + m_i \tilde{Q}_i Q^i, \quad m_i \rightarrow 0$$

are known ^{20,21}. (See Table).

Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f)$
monopoles	$U(1)^{n_c-1}$	Confinement	$U(n_f-1) \times U(1)$
NA monopoles	$SU(r) \times U(1)^{n_c-r}$	Confinement	$U(n_f-r) \times U(r)$
rel. nonloc.	-	Confinement	$U(n_f/2) \times U(n_f/2)$
NA monopoles	$SU(\tilde{n}_c) \times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$U(n_f)$

Deg.Freed.	Eff. Gauge Group	Phase	Global Symmetry
rel. nonloc.	-	Confinement	$U(n_f)$
dual quarks	$USp(2\tilde{n}_c) \times U(1)^{n_c-\tilde{n}_c}$	Free Magnetic	$SO(2n_f)$

From these results we learn, in particular, that the spectrum of the “dual quarks” in the infrared theory (charges, multiplicity, flavor) is identical to what is expected from the semiclassical abelian or nonabelian monopoles. We note in particular that the r - vacua (*i.e.* vacua with a low-energy effective $SU(r)$ gauge group) exist only for $r < \frac{n_f}{2}$, namely as long as the sign-flip of the beta function occurs:

$$b_0^{(dual)} \propto -2r + n_f > 0, \quad b_0 \propto -2n_c + n_f < 0.$$

Indeed, analogous r vacua exist semiclassically for all values up to $\min(n_f, n_c)$, but quantum mechanically, only those with $r \leq n_f/2$ give rise to vacua with nonabelian gauge symmetry. Also, when the sign flip is not possible (e.g. $N = 2$ YM or on a generic point of the quantum moduli space) dynamical Abelianization is expected and does take place!

These observations led us to conclude that the “dual quarks” belonging to the fundamental representation of the infrared $SU(r)$ gauge group,

actually *are* the Goddard-Nuyts-Olive-Weinberg monopoles, which have become massless by quantum effects ²².

Most importantly, we are led to the general criterion for nonabelian monopoles to survive quantum effects: the system must produce, upon symmetry breaking, a sufficient number of massless flavors to protect H from becoming too strongly-coupled. Natural embedding in $N = 2$ systems for various cases in Table 1 has been discussed in Ref.¹⁹

A very subtle hint about the nature of the nonabelian monopoles come from the recent discovery of nonabelian vortices.

9. Vortices

Vortices occur in a system where a gauge group H is broken to some discrete group

$$H \xrightarrow{\langle \phi \rangle \neq 0} C$$

such that $\Pi_1(H/C)$ is not trivial. Gauge field behaves far from the vortex axis as

$$A_i \sim \frac{i}{g} U(\varphi) \partial_i U^\dagger(\varphi); \quad \phi_A \sim U \phi_A^{(0)} U^\dagger, \quad U(\varphi) = \exp i \sum_j^r \beta_j T_j \varphi$$

Quantization condition reads $\alpha \cdot \beta \in \mathbb{Z}$ where β_i are weight vectors of \tilde{H} , dual of H . Some known cases are:

- $H = U(1)$: in this case vortices correspond to the well-known Abrikosov-Nielsen-Olesen vortices, representing elements of $\Pi_1(U(1)) = \mathbb{Z}$. According to the parameters appearing in the system they yield Type I, Type II or BPS superconductors;
- The case $H = SU(N)/\mathbb{Z}_N$ yields \mathbb{Z}_N vortices. These are non BPS and are difficult to analyse (model dependence), although there are some interesting work on the tension ratios, the sine formula ($T_k \propto \sin \frac{\pi k}{N}$), etc.²³

10. Nonabelian Vortices

Truely nonabelian vortices (*i.e.*, with a nonabelian flux) have recently been constructed ^{24,25}. In the simplest case, we consider the system

$$SU(3) \xrightarrow{v_1} \frac{SU(2) \times U(1)}{\mathbb{Z}_2} \xrightarrow{v_2} 0, \quad v_1 \gg v_2,$$

The high-energy theory has monopoles; the low-energy theory (monopoles heavy) has vortices. We are here mainly interested in the low-energy theory ($\frac{SU(2) \times U(1)}{\mathbb{Z}_2} \xrightarrow{v_2} 0$). We embed the system in a $N = 2$ model with number of flavor, $4 \leq n_f \leq 5$, so as to maintain the “unbroken” subgroup $SU(2)$ non asymptotically free. We shall take the bare mass m and the adjoint scalar mass $\mu \Phi^2$, so that $v_2 = \xi = \sqrt{\mu m} \ll v_1 = m$. The scalar VEVs are

$$\Phi = -\frac{1}{\sqrt{2}} \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & -2m \end{pmatrix}, \quad \langle q^{kA} \rangle = \langle \tilde{q}^{kA} \rangle = \sqrt{\frac{\xi}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (4)$$

where only nonvanishing color (vertical) and flavor (horizontal) components of squarks are shown. Set $\Phi = \langle \Phi \rangle$; $q = \tilde{q}^\dagger$; and $q \rightarrow \frac{1}{2}q$, then the action density is

$$\left[\frac{1}{4g_2^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2} (F_{\mu\nu}^8)^2 + |\nabla_\mu q^A|^2 + \frac{g_2^2}{8} (\bar{q}_A \tau^a q^A)^2 + \frac{g_1^2}{24} (\bar{q}_A q^A - 2\xi)^2 \right].$$

11. Nonabelian Bogomolnyi Equations

The tension reads

$$T \int d^2x \left[\sum_{a=1}^3 A_a^2 + B^2 + \frac{1}{2}|C|^2 + \frac{\xi}{2\sqrt{3}} \tilde{F}^{(8)} \right],$$

$$A_a = \frac{1}{2g_2} F_{ij}^{(a)} \pm \frac{g_2}{4} (\bar{q}_A \tau^a q^A) \epsilon_{ij}, \quad B = \frac{1}{2g_1} F_{ij}^{(8)} \pm \frac{g_1}{4\sqrt{3}} (|q^A|^2 - 2\xi) \epsilon_{ij},$$

$$C = \nabla_i q^A \pm i\epsilon_{ij} \nabla_j q^A,$$

leading to the nonabelian Bogomolnyi Equations, $A_a = B = C = 0$. The vortex flux ($SU(N)$) is

$$\vec{B} = \nabla \wedge \vec{A}, \quad F_v = \int_{R^2} d\mathbf{S} \cdot \frac{\text{Tr } \phi \mathbf{B}}{\frac{1}{\sqrt{2}} (\text{Tr } \phi \phi)^{1/2}} = 2\pi \cdot \sqrt{\frac{2(N+1)}{N}},$$

This matches exactly the monopole flux Eq. (3)²⁸. A crucial fact is that there is an unbroken global symmetry, $SU(2)_{C+F}$ (see Eq. (4)), broken only by the vortex configuration (to a $U(1)$). This implies the existence of exact zero-modes (moduli) labelling

$$SU(2)/U(1) = S^2 = \mathbf{CP}^1.$$

The vortex of Generic Orientation (Zero Modes) can be explicitly constructed as

$$q^{kA} = U \begin{pmatrix} e^{i\varphi} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = e^{\frac{i}{2}\varphi(1+n^a\tau^a)} U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1},$$

$$\begin{aligned}\mathbf{A}_i(x) &= U \left[-\frac{\tau^3}{2} \epsilon_{ij} \frac{x_j}{r^2} [1 - f_3(r)] \right] U^{-1} = -\frac{1}{2} n^a \tau^a \epsilon_{ij} \frac{x_j}{r^2} [1 - f_3(r)], \\ A_i^8(x) &= -\sqrt{3} \epsilon_{ij} \frac{x_j}{r^2} [1 - f_8(r)]\end{aligned}$$

where

$$U \in SU(2)_{C+F}, \quad U \tau^3 U^\dagger = n^a \tau^a,$$

$$n^a = (\sin \alpha \cos \beta, \sin \alpha \sin \beta, \cos \alpha), \quad U = e^{-i\beta \tau_3/2} e^{-i\alpha \tau_2/2}.$$

The tension $T = 2\pi\xi$ is independent of U .

Remarks:

In more general $SU(N+1) \rightarrow \frac{SU(N) \times U(1)}{\mathbb{Z}_N} \rightarrow \emptyset$ systems with flavors $2N+2 > N_f \geq 2N$, there appear vortices with $2(N-1)$ - parameter family of zero modes, parametrizing

$$\frac{SU(N)}{SU(N-1) \times U(1)} \sim \mathbb{CP}^{N-1} :$$

they nicely match the space of (quantum) states of a particle in the fundamental representation of an $SU(N)$. (Actually, for $N_f > N$ there are other vortex zero modes (semilocal strings), not related to the unbroken, exact $SU(N)_{C+F}$ symmetry. Those are related to the flat directions.)

Furthermore, vortex dynamics ($\frac{SU(2) \times U(1)}{\mathbb{Z}_2} \rightarrow \emptyset$)

$$\mathbf{n} \rightarrow \mathbf{n}(z, t)$$

can be shown to be equivalent to:

$$S_\sigma^{(1+1)} = \beta \int d^2x \frac{1}{2} (\partial n^a)^2 + \text{fermions} :$$

an $O(3) = \mathbb{CP}^1$ sigma model^{26,27}! It has two vacua; no spontaneous breaking of $SU(2)_{C+F}$ occurs; Also, there is a close connection between the 2D vortex sigma model dynamics and the 4D gauge theory dynamics: they are dual to each other²⁹.

In $N = 2$ theory, due to the presence of two independent scales which we take very different ($\mu \ll m$), we can study monopoles (HE theory) and vortices (LE theory) separately in the effective theories valid at respective scales. Physically, of course, it is perfectly sensible to consider both together; the only problem is that it becomes very difficult to disentangle the two if the two scales are of the same order.

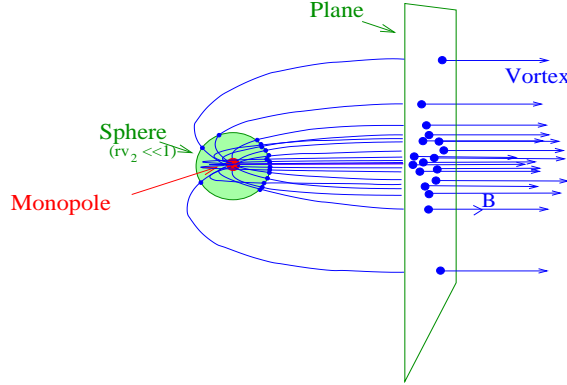
Nevertheless, there remains the fact that monopoles (of the HE theory) and vortices (of the LE theory) are actually incompatible

- as static configurations. In fact, the monopoles of HE theory represent $\Pi_2(SU(N+1)/SU(N) \times U(1)/\mathbb{Z}_N)$; but in the full theory, $\Pi_2(SU(N+1)) = \emptyset$, so no topologically stable monopoles exist. On the other hand, vortices represent $\Pi_1(\frac{SU(N) \times U(1)}{\mathbb{Z}_N})$ of LE theory; no vortices exist in the full theory since $\Pi_1(SU(N+1)) = \emptyset$.

What happens is that monopoles of G/H are confined by magnetic vortices of $H \rightarrow \emptyset$, leading to monopole-vortex-antimonopole bound states, which are not stable as static configurations. They could however give rise to rotating and dynamically stable states. After all, the mesons in QCD *are* systems of this kind!

Note that the restriction on the number of flavors, $2N+2 > N_f \geq 2N$, is fundamental. If N_f were less than $2N$, the subgroup $SU(N)$ would become strongly coupled, and break itself dynamically. Nonabelian vortices do not exist quantum mechanically in such a system.

We are then led to the following relation between the vortex zero-modes and \tilde{H} transformation of monopoles. Consider a configuration consisting of a monopole (of G/H) and an infinitely long vortex, which carries away the full monopole flux. At small distances r from the monopole center, $r \sim O(1/m)$, HE theory is a good approximation and the monopole flux looks isotropic; at a much larger distances of order of, $r > \frac{1}{\xi}$, one sees the vortex of LE theory. The energy of the configuration is unchanged if the whole system is rotated by the exact H_{C+F} transformation. This is a nonlocal transformation. The end point monopole is apparently transformed by the H_C part only, but, since in order to keep the energy of the whole system unchanged it is necessary to transform the whole system, it is not a simple gauge transformation H of the original theory. It is in this sense that the nonlocal, global H_{C+F} transformations can be interpreted as the dual transformation \tilde{H} acting on the monopole, at the endpoint of the vortex.



12. To conclude: where do we stand ?

- Nonabelian monopoles are intrinsically quantum mechanical;
- Massless flavors are important for (i) keeping H unbroken; and (ii) for providing enough global symmetry giving rise to exact vortex zero-modes: these can be interpreted as the dual gauge transformation acting on the monopoles at the ends of the vortex;
- One has a nice "model" of monopole confinement by vortices.
- Light nonabelian monopoles appear as IR degrees of freedom (examples in $N = 2$ models). Are there light nonabelian monopoles in some other $N = 1$ theories?
- Do some vacua of $N = 2$ theories, especially those based on "almost superconformal vacua" ³⁰ provide a good model of confinement in QCD?

Acknowledgment

I thank Arkady Vainshtein and the organizers of the Workshop for their kind hospitality and for providing us with an occasion for interesting discussions with many participants.

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